Section 1.1

Tangent line problem example: Find the slope of a secant line for the function $f(x) = x^2$. Since $f(1) = 1$ and $f(2) = 4$, we know that the points $P(1,1)$ and $Q_0(2,4)$ lie on the graph of $f(x)$. Following the formula

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

we can find the slope of the secant line between the points $P(1,1)$ and $Q(2,4)$. Do you see that in this case $c = 1$ and $\Delta x = 1$? The slope of this secant line is then

$$m_{sec} = \frac{f(2) - f(1)}{1} = 4 - 1 = 3$$

1) Let $f(x) = x^2$. In the problem, first identify c and then in a – e identify Δx.
   Find the slope of the secant line between the points $P(1,1)$ and

   a) $Q_1\big(1.5, f(1.5)\big)$

   b) $Q_2\big(1.1, f(1.1)\big)$

   c) $Q_3\big(1.01, f(1.01)\big)$

   d) $Q_4\big(1.001, f(1.001)\big)$

   e) $Q_5\big(1.0001, f(1.0001)\big)$

   f) What do you notice about the answers in parts a – e?

   g) What is the slope of the tangent line for the function $f(x) = x^2$ at the point $P(1,1)$?

In the previous problem, notice that in order to find the slope of the tangent line at the point $P(1,1)$ for the function $f(x) = x^2$ we found the slopes of several secant lines, with each point Q_i closer and closer to P. We then noticed that the slopes of each of these secant lines appeared to be getting closer and closer to 2 as Δx got smaller and smaller. This is idea of a limit.

Area problem example: Consider the region bounded by the function $f(x) = x^2$, the x-axis, the line $x = 1$, and the y-axis (it might help you to draw this region).

We can approximate this region with rectangles by dividing it into four parts. If we start at $x = 0$, then we can evaluate the function at $x = 0, x = \frac{1}{4}, x = \frac{1}{2}$, and $x = \frac{3}{4}$ (notice that this means all of the approximating rectangles have width $\Delta x = \frac{1}{4}$).

This is called a left-endpoint approximation since we are evaluating the function at the left endpoints of each approximating rectangle. Since $f(0) = 0$, the area of the first approximating rectangle is $A_1 = 0$.

The area of the second approximating rectangle is $A_2 = f\left(\frac{1}{4}\right) \cdot \Delta x = \left(\frac{1}{16}\right)\left(\frac{1}{4}\right) = \frac{1}{64}$.

Similarly, we get $A_3 = f\left(\frac{1}{2}\right) \cdot \Delta x = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$, and $A_4 = f\left(\frac{3}{4}\right) \cdot \Delta x = \left(\frac{9}{16}\right)\left(\frac{1}{4}\right) = \frac{9}{64}$.

Adding the areas of all of the approximating rectangles, we get an approximate area for the given region:

$$A_{\text{lower}} = A_1 + A_2 + A_3 + A_4 = 0 + \frac{1}{64} + \frac{1}{16} + \frac{9}{64} = \frac{7}{32} = 0.21875$$

Another way of approximating the given region is to use a right-endpoint approximation, where we start at $x = \frac{1}{4}$ and again count up by $\Delta x = \frac{1}{4}$ until we reach $x = 1$.

Using this approximation method, we get $A_1 = \frac{1}{64}, A_2 = \frac{1}{16}, A_3 = \frac{9}{64}$, and $A_4 = \frac{1}{4}$, so the total area of the approximating rectangles in this case is

$$A_{\text{upper}} = A_1 + A_2 + A_3 + A_4 = \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} = \frac{15}{32} = 0.46875$$

Since our first method gave an under-approximation of the actual area and the second method gave an over-approximation, we can conclude that actual area is somewhere between $\frac{7}{32}$ and $\frac{15}{32}$, or

$$\frac{7}{32} < A_{\text{actual}} < \frac{15}{32}$$

2) Using the methods outlined above you will now divide the same region into five equal rectangles and find a lower and upper approximation to the area under the curve $f(x) = x^2$ from $x = 0$ to $x = 1$.

   a) Draw a picture of the given region with the area divided into five rectangles of equal width, starting at $x = 0$. This will be a left-endpoint approximation.

   b) Find the areas of each of the approximating rectangles, and then add up the areas to find a lower approximation of the area of the region.

   c) Draw a picture of the given region with the area divided into five rectangles of equal width, starting at $x = 0$, but with the height of the first rectangle found by evaluating the function at $x = \frac{1}{5}$. This will be a right-endpoint approximation.

   d) Find the areas of each of the approximating rectangles, and then add up the areas to find an upper approximation of the area of the region.

Notice that the approximations you got in the previous problem gave a narrower range for what the actual area could be than when we used four rectangles. This leads us to conclude that if we were to use even more rectangles to approximate the area, we would get closer to the actual area. If we let the width of each of the rectangles approach zero (this means the number of rectangles we use approaches infinity), then we can find the actual area of the region. This is the idea of a limit.

Homework for this section: Read section 1.1. Watch the videos (marked with  in the e-book) and do the tutorials (marked with  in the e-book). Do problems 6 and 7 on p. 47. Come to class with at least two questions related to what you read/watched.